Directed theories of time and the conventionality of simultaneity

Abstract. This paper shows a sense in which the non-conventionality of simultaneity extends to uniformly accelerating observers. I argue that this provides evidence against directed theories of time.

1.

David Malament (1977) argued that, in the framework of special relativity, the relation of *simultaneity relative to a uniform-velocity observer*[1] is more than a mere convention: on his reading, the standard relation is provably unique. In this article, I will argue that:

(1) Malament's result can be extended to uniform-*acceleration* observers; and
(2) this extension shows a sense in which directed theories of time are false.

A 'directed' theory of time is one that distinguishes an objective future-directed time ordering for physical processes, where I take 'objective' to mean that the order is independent of the judgements of co-directed observers.[2] In the philosophy of time discussion set out by McTaggart (1908), directedness is a feature of both the 'B-series' and the 'A-series', as compared to the undirected 'C-series', which remain extremely popular. Of course, they are not without their critics. For example, the work of Price (1996) led to a collection of arguments against directed theories of time. I am sympathetic to these arguments. But, in this article, I will introduce a critique of a different sort.

I begin with an example of how co-directed observers in special relativity can disagree about the order of processes in time, undermining the possibility of an objective future direction of time in the sense above. The example turns on the judgements of uniform-acceleration observers, equipped with a particular simultaneity relation. I prove that this is the unique simultaneity relation available to such observers, in a sense very similar to the one established by Malament for uniform-velocity observers. I conclude by discussing this result in light of some critical responses to Malament.

2.

There is a well-known way to visualise the standard simultaneity relation in special relativity, so let me begin by making my point in a non-technical way. Suppose we adopt a coordinate system for a uniform velocity observer in which the observer

[1]In some literatures this is also referred to as *clock synchronicity* for an inertial observer.

[2]Directed theories of time don't stand a chance if we open up our standard of objectivity to include observers that are not necessarily co-directed. (Timelike curves in Minkowski spacetime with tangent fields ξ^a and χ^a are *co-directed* if and only if $\chi_a\xi^a > 0$.)

is at 'rest', and the observer's worldline is directed 'straight up'. The standard simultaneity relation says: two simultaneous events for this observer will lie on a surface (called a *simultaneity surface*) orthogonal to this worldline. The surfaces are illustrated in the centre image of Figure 1. In contrast, for an observer moving with uniform velocity to the left or right in this coordinate system, the simultaneity surfaces will be 'tilted up' in the direction of motion, to a degree above the horizontal equal to the degree that the worldline tilts from the vertical. The simultaneity surfaces for these observers are illustrated in the left and right images of Figure 1, respectively. In special relativity, observers with different uniform velocities will generally disagree about which spacelike-separated events are simultaneous; however, all co-directed observers agree on the ordering of physical processes, at least as represented by a sequence of timelike-separated events.

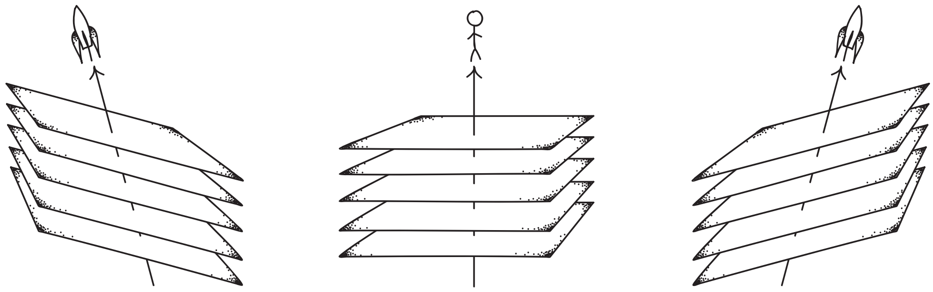

FIGURE 1. Simultaneity surfaces for uniform velocity observers.

The simultaneity surfaces for each of these three observers are orthogonal to their worldlines: not in the Euclidean metric, but in the Minkowski metric of special relativity.[3] And, it is these surfaces that define the equivalence classes of the simultaneity relation inn Malament's uniqueness theorem: simultaneous points for a uniform-velocity observer lie on the surface that is orthogonal to that observer's worldline.

Malament's theorem assumes the observer's worldline is a straight line associated with uniform-velocity motion. However, the orthogonality of a surface and a line does not require that line to be straight: orthogonality is defined at each point on an arbitrary curve by way of its tangent vector field. So, this simultaneity relation can be extended to accelerating observers as well. Reserving most formal details for the next section, let me illustrate how this works with another picture.

Consider an observer who moves with constant acceleration, slowing down in one direction until coming to rest, and then turning around and speeding back up in the reverse direction. The surfaces that are orthogonal to this observer will 'tilt' to a different degree as the observer's instantaneous velocity changes. As it turns out, they all intersect on a plane O, as shown in Figure 2. But if one views these

[3]That is, in Minkowski spacetime $(\mathbb{R}^4, \eta_{ab})$, the standard simultaneity relation for an inertial line L is defined by the surfaces orthogonal to L, in that the tangent field ξ^a for L and a tangent field χ^a for the surface are orthogonal in the Minkowski metric at their point of intersection, $\xi^a\chi_b = \eta_{ab}\xi^a\chi^b = 0$.

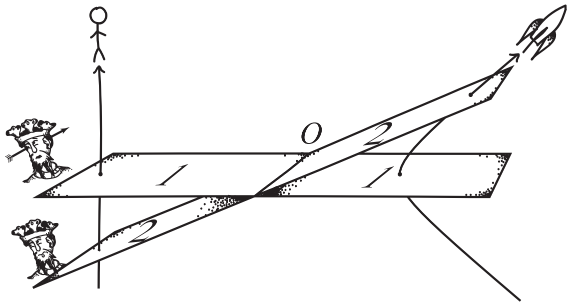

FIGURE 2. Reverse chronology for an accelerating observer

orthogonal surfaces as simultaneity surfaces, then one finds that they determine a curious chronology. Two timelike-separated events to the left of the surface O, say, 'Harold is crowned King', and 'Harold is slain', will be judged by an inertial observer to occur one before the other. But, the co-directed accelerated observer will judge these events to occur in the reverse order. That is, insofar as we have chosen the correct simultaneity surfaces for each observer (the orthogonal ones), *co-directed observers disagree about the order of this causal process.* We thus face the following argument.

1. If the simultaneity surfaces for accelerated observers are the orthogonal ones, then co-directed observers disagree about the order of timelike-separated events.
2. If co-directed observers disagree about the order of timelike-separated events, then directed theories of time are false.
3. The simultaneity surfaces for accelerated observers are the orthogonal ones.

C. Directed theories of time are false.

The first premise is established by our example, while the second premise expresses the definition of a 'directed' theory of time. So, the conclusion of this argument hinges on the status of the third premise. How does it fare?

The simultaneity surfaces of the accelerating observer have two strange properties. First, they do not cover spacetime, and so there are events for which the accelerated observer will can make no simultaneity judgements at all. Second, they intersect on the plane[4] O, which means that the accelerating observer would assign multiple different times to the same events in this region. Both of these features challenge the coherence of this observer's judgements, and might be taken as reason to reject the third premise.

This would be too quick. When the (measure zero) plane of intersection O is removed from consideration, and when we restrict attention to the region outside the light cone that the accelerated observer approaches, then the problems dissolve. This

[4] In Figure 2, I have illustrated two-dimensional spatial surfaces, and their region of intersection is a one-dimensional line. The corresponding (more realistic) three-dimensional spatial surfaces will intersect on a two-dimensional plane. I beg the reader's forgiveness for my failure to draw this case.

region is sometimes referred to in the physics literature as the (left and right) 'Rindler wedge' region, without the central plane O. For short, I will refer to it as *Rindler region* R_O; it consists in the left and right wedge regions with O removed, shown in the two-dimensional diagram of Figure 3. Restriction to this region is common in considerations of mathematical physics like the Unruh effect[5]. But, more importantly for this discussion, this seems to me to be the only region that has any hope of supporting a simultaneity relation for an accelerating observer at all. So, it makes sense to restrict attention to this when seeking a definition of simultaneity of any kind for observers.

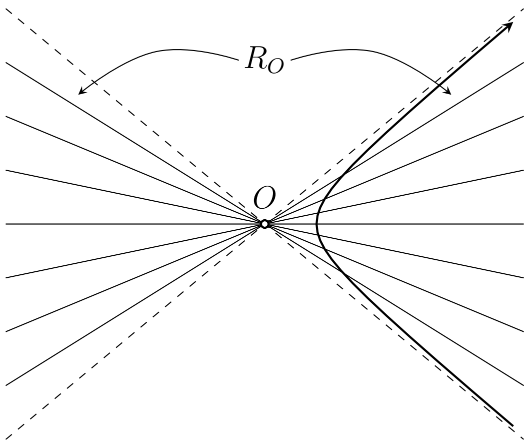

FIGURE 3. Accelerated simultaneity surfaces in the Rindler region R_O.

In the next section, I will show that such a simultaneity relation is indeed available in this region. Not only does the standard (orthogonality) simultaneity relation exist for an accelerated observer in the Rindler region, but it is provably unique, in the same sense of uniqueness that Malament established for an inertial observer's simultaneity relation. Therefore: insofar as this result establishes our third premise above, it establishes the falsity of directed theories of time.

3.

Let me now turn to the existence and uniqueness of an accelerating observer's simultaneity relation in the Rindler region R_O. This section will be (unavoidably) technical in nature. However, it is a straightforward extension of the theorem established in Malament (1977), which is itself motivated by the work of Robb (1914). My argument closely follows Malament's unpublished notes on simultaneity for uniform-velocity observers (Malament; 2009, §3.4). I begin with a generalisation of the standard simultaneity relation for uniform-velocity observers, to observers following an arbitrary timelike worldline.

[5]See e.g. the discussion of Earman (2011).

Definition 1 (Sim_C)**.** Let C be any smooth timelike curve in Minkowski spacetime $(\mathbb{R}^4, \eta_{ab})$. The *simultaneity relation* Sim_C is the set of pairs $(p, q) \in \mathbb{R}^4 \times \mathbb{R}^4$ for which there exists a (spatially) flat hypersurface $\Sigma \subset \mathbb{R}^4$ that intersects C at a point, is orthogonal[6] there, and $p, q, \in \Sigma$.

For arbitrary curves, Sim_C is not an equivalence relation. But, it is for uniform-velocity (inertial) observers, where it defines the standard simultaneity relation derived by Malament (1977). And, it is easy to see that it is for a uniformly accelerated observer as well, in the Rindler region $\mathcal{R}_O$ consisting of the right and left Rindler wedges without the surface of intersection O. We can state this formally as:

Proposition 1 (Existence)**.** Let C be a smooth, uniformly accelerating[7] timelike curve in the Rindler region $\mathcal{R}_O \subset \mathbb{R}^4$ (without centre O). Then:

(S1) Sim_C is an equivalence relation (reflexive, symmetric, transitive) on points $(p, q) \in \mathcal{R}_O \times \mathcal{R}_O$; and
(S2) For all $p \in \mathcal{R}_O$, there exists a unique $q \in C$ such that $(p, q) \in \mathrm{Sim}_C$.

The proof follows Malament exactly, and can be read off the diagram of Figure 3. So, let us turn to the uniqueness result. We begin with:

Definition 2 (C-isometry, C-invariance)**.** Let C be a smooth curve in Minkowski spacetime $(\mathbb{R}^4, \eta_{ab})$. A *$C$-isometry* is an isometry $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$ that preserves C, in that $p \in C$ only if $\varphi(p) \in C$.

A two-place relation S on a region $U \subseteq \mathbb{R}^4$ that is *C-invariant* is one that is preserved by C-isometries, in that $(p, q) \in S$ if and only if $\big(\varphi(p), \varphi(q)\big) \in S$, for all C-isometries φ and for all $p, q \in U$.

We will be concerned with two kinds of isometries: the Lorentz boosts φ_L, which translate along the invariant hyperbolas of Minkowski spacetime, and time-reflection about a spatial surface. They are both required in the proof of the uniqueness result:

Proposition 2 (Uniqueness)**.** Let C be a smooth, uniformly accelerating timelike curve in the Rindler region $\mathcal{R}_O$ (without centre O), and let S be a two-place relation on $\mathcal{R}_O$. If S satisfies (S1) and (S2), and is also C-invariant, then $S = \mathrm{Sim}_C$.

Proof. For all $p \in \mathcal{R}_O$, let $c(p)$ be the (unique) point in the flat hypersurface Σ containing p such that $c(p) \in C$. Then $\big(p, c(p)\big) \in \mathrm{Sim}_C$ for all $p \in \mathcal{R}_O$, and also $(p, q) \in \mathrm{Sim}_C$ iff $c(p) = c(q)$ for all $p, q \in \mathcal{R}_O$.

The main step of the proof is to show that the following condition also holds:

$$\big(p, c(p)\big) \in S \text{ for all } p \in \mathcal{R}_O. \tag{1}$$

[6]i.e. if ξ^a and χ^a are respective tangent vector fields for Σ and C, then $\chi_a \xi^a = 0$ at the point of intersection.

[7]i.e. the acceleration of its tangent vector field $\xi^n \nabla_n \xi^a$ has constant norm and is everywhere parallel to itself.

This quickly entails our conclusion, following Malament's argument exactly: first, ($\Rightarrow$) assume $(p,q) \in S$. Then $\big(q, c(q)\big) \in S$ by Condition (1), and so by transitivity (S1) we have $\big(p, c(q)\big) \in S$. But $\big(p, c(p)\big) \in S$ as well, and so by the uniqueness statement of (S2), we have that $c(p) = c(q)$. This implies that $(p,q) \in \mathrm{Sim}_C$. Conversely, ($\Leftarrow$) assume $(p,q) \in \mathrm{Sim}_C$. Then $c(p) = c(q)$. Moreover, by Condition (1), $\big(p, c(p)\big) \in S$, and thus $\big(p, c(q)\big) \in S$. Therefore, since $\big(q, c(q)\big) \in S$ by (S2), we have by reflexivity and transitivity that $(p,q) \in S$.

We thus turn to establishing Condition (1): let $p \in \mathcal{R}_O$. By (S2), there is a unique point $q \in C$ such that $(p,q) \in S$. For a uniformly accelerating curve, there is one temporal reflection φ_T that is a C-isometry. We write Σ to denote the surface about which it reflects. Let φ_L be the Lorentz boost that maps p onto Σ; it also maps $c(p)$ onto Σ, and is a C-isometry.

Since $\varphi_L(p)$ is on the reflection surface, the C-isometries φ_L and φ_T have the property that $\varphi_T\big(\varphi_L(p)\big) = \varphi_L(p)$, or equivalently: $\Phi(p) = p$, where $\Phi = \varphi_L^{-1} \circ \varphi_T \circ \varphi_L$. Moreover, by the C-invariance of S, $(p,q) \in S$ implies that,

$$(\Phi(p), \Phi(q)) \in S.$$

So, $(p, \Phi(q)) \in S$. But both q and $\Phi(q)$ are on the curve C, so by the uniqueness clause of (S2), $q = \Phi(q)$, or equivalently, $\varphi_L(q) = \varphi_T \circ \varphi_L(q)$.

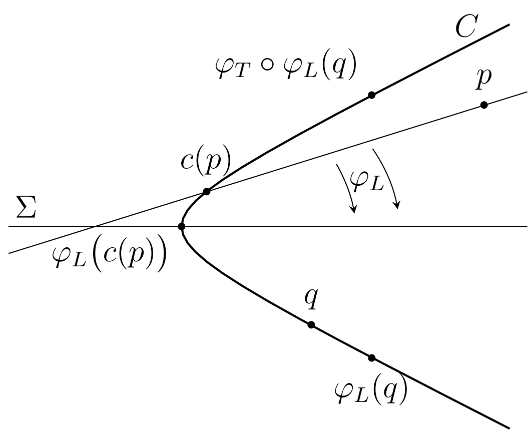

FIGURE 4.

Now, since $\varphi_L\big(c(p)\big)$ is on the reflection surface Σ and on C, it must be between $\varphi_L(q)$ and $\varphi_T \circ \varphi_L(q)$; see Figure 4. So, $\varphi_L(q) = \varphi_T \circ \varphi_L(q)$ implies that $\varphi_L(q) = \varphi_L\big(c(p)\big)$. Applying φ_L^{-1} to both sides thus gives $q = c(p)$, which implies $\big(p, c(p)\big) \in S$, and hence Condition (1). $\square$

4.

Some of the critical responses to Malament's assumptions apply to the present argument as well. I have little to add to that debate.[8] But I will address one pressing concern, that Malament's original argument (and mine, so far) rely on an assumption of time reversal symmetry.[9] The uniqueness results we prove require that the simultaneity relation S be invariant under the isometries that preserve a curve C — including time reversal. This ensures that the phrase 'simultaneous events' means the same thing in all regions that are related by a symmetry of Minkowski spacetime structure, such as those that differ by a spatial translation, or by a change of temporal direction. But this latter assumption can be questioned. The assumption that the meaning of simultaneity is invariant under time reversal might be viewed as begging the question against defenders of a directed theory of time, who simply assume the opposite. Moreover, from a scientific perspective, it is well-known that certain weakly-interacting matter fields violate time reversal symmetry. And, without the assumption of time reversal invariance, the uniqueness result fails: there are many simultaneity relations S that are preserved by the time-reversal-excluding C-isometries of a uniformly accelerating curve, including the 'bent surfaces' of Figure 5. This kind of simultaneity relation is invariant under (Lorentz) velocity boosts, but not under time reversal.

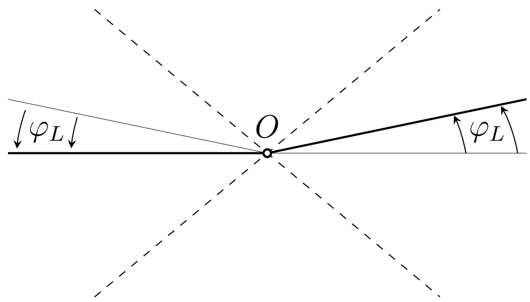

FIGURE 5. 'Bent surfaces' of simultaneity related by a boost φ_L

[8]Redhead (1993, p.114) suggests that in order to avoid the result, "we can dispute that simultaneity need be an equivalence relation", or adopt a privileged "neo-Lorentzian" frame of reference. But we are interested in situations where simultaneity has some well-defined meaning, whereas it is not obvious what that would be if it is not an equivalence relation. And, a "neo-Lorentzian" approach to relativity theory is arguably a different theory entirely, and the status of simultaneity still remains of interest in standard (Minkowski spacetime) approaches. Janis (1983, 2018) argues that, by allowing for the conventional choice of an initial curve representing the observer, the conventionality of simultaneity is restored. But this simply shifts the meaning of "conventionality of simultaneity" (in a single reference frame) to be nothing more than the well-known "relativity of simultaneity" (relative to different reference frames), which is a well-understood fact about Minkowski spacetime.
[9]This issue was pointed out by Sarkar and Stachel (1999) and Ben-Yami (2006), and is discussed by Malament (2009, p.61).

The uniqueness results are of independent interest in the (time reversible) cases in which they apply. But more importantly, this objection simply does not solve the problem for the defender of a directed theory of time. Notice that the counterexample above still features chronology reversal, in that 'moving up' through such simultaneity surfaces on the right wedge corresponds to 'moving down' on the left. This suggests that the problem actually arises out of facts about the Lorentz transformations of special relativity, and has nothing to do with time reversal. Indeed, it is possible to prove a precise version of this statement. To give it, we first need a definition of isometry and invariance that excludes time reversal. Let τ^a be a smooth timelike vector field in a temporal orientation[10]; this allows us to define a *future-directed* vector field ξ^a to be one satisfying $\tau^a \xi_a > 0$.

Definition 3 ($C^{\uparrow}$-isometry, $C^{\uparrow}$-invariance)**.** Let $(\mathbb{R}^4, \eta_{ab}, \tau^a)$ Minkowski spacetime with a temporal orientation. Let C be a smooth curve in Minkowski spacetime $(\mathbb{R}^4, \eta_{ab})$. A $C^{\uparrow}$-*isometry* is a C-isometry that preserves the temporal orientation. A two-place relation S on $\mathbb{R}^4$ that is $C^{\uparrow}$-*invariant* is one that is preserved by all $C^{\uparrow}$-isometries.

We will also need to define what it means for a simultaneity relation to respect (and not respect) chronology. Let I denote an interval of real numbers, to serve as indices for a time.

Definition 4 (chronology-respecting relation)**.** Let $(\mathbb{R}^4, \eta_{ab}, \tau^a)$ Minkowski spacetime with a temporal orientation. A two-place equivalence relation S on $U \subseteq \mathbb{R}^4$ is *chronology-respecting* if and only if the equivalence classes of S can be ordered as a one-parameter family of sets $\{\Sigma_t \mid t \in I\}$ where, if f is the function that maps each $p \in U$ to the index t for which $p \in \Sigma_t$, then f is a global time function[11] for U.

The problem for directed theories of time can now be considerably strengthened. Unlike Malament's original argument, we do not need to assume time reversal invariance, or even (S2), that for all $p \in \mathcal{R}_O$, there exists a unique $q \in C$ such that $(p, q) \in S$. If the simultaneity relation S is an equivalence relation that is invariant under the non-time-reversing symmetries that preserve a curve C, then it cannot be chronology-respecting in the way that directed theories of time require.

Proposition 3 (No Rindler Chronology)**.** Let $(\mathbb{R}^4, \eta_{ab}, \tau^a)$ be Minkowski spacetime with a temporal orientation. Let C be a smooth, uniformly accelerating timelike curve in the Rindler region $\mathcal{R}_O$ (without centre O), and let S be a two-place relation

[10] A *temporal orientation* on a Lorentzian manifold (M, g_{ab}) is an equivalence class of all smooth timelike vector fields $[\tau^a]$ that are co-directed. If such an equivalence class exists, then there are exactly two, consisting of the vector fields that all 'point' into the same lightcone lobe, providing a global way to distinguish the past and future at every spacetime point.

[11] A *global time function* on U is one whose derivative $\nabla^a f$ is a smooth future-directed timelike vector field.

on $\mathcal{R}_O$. If S is an equivalence relation (S1) and also $C^{\uparrow}$-invariant, then it is not chronology-respecting.

Proof. Suppose for *reductio* that the opposite is true: let $\{\Sigma_t \mid t \in I\}$ be the chronology-respecting family of equivalence classes for S, and let f be the associated global time function. This f satisfies $\xi_a \nabla^a f > 0$ for every future-directed timelike vector ξ^a. So, $\xi(f) > 0$, which implies that f increases monotonically on future-directed timelike curves.

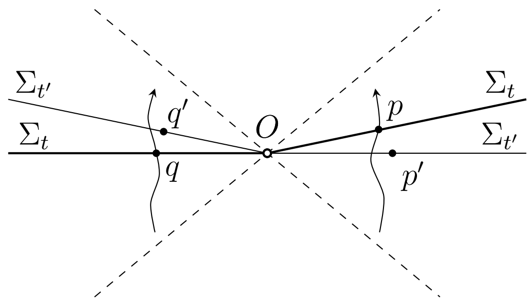

FIGURE 6.

But f cannot be monotonically increasing on such curves in both wedges of $\mathcal{R}_O$. To see this, let φ_L be a C-preserving Lorentz boost that is not the identity. Then, for any equivalence class Σ_t, $C^{\uparrow}$-invariance implies that $\varphi_L(\Sigma_t)$ is also an equivalence class, which we denote $\Sigma_{t'}$ for some $t' \neq t$. Consider a future-directed timelike curve that intersects Σ_t and $\Sigma_{t'}$ and lies in the same wedge as C. Then the point p where the curve intersects Σ_t is in the causal future of the point $p' := \varphi_L(p) \in \Sigma_{t'}$. The opposite is true for a future-directed timelike curve in the wedge opposite of C for points $q \in \Sigma_t$ and $q' \in \Sigma_{t'}$ (Figure 6). So, if $f(p) = t < t' = f(p')$ in one wedge, then $f(q) = t > t' = f(q')$ in the other, which is a contradiction. □

As a last resort, the defender of a directed theory of time might deny that an observer will view simultaneity surfaces as relevant for determining the direction of time.[12] But this would be to reject one of the following two assumptions, both of which seem plausible (to me). Suppose we assume, as is ubiquitous in special relativity, that an observer,

(1) uses a simultaneity surface to judge which points occur simultaneously at a given event on their worldline; and,
(2) experiences events as a directed temporal ordering, at least in some psychological sense.

[12]I thank Adam Caulton for this suggestion (private communication).

Then the observer's experience of simultaneity surfaces will have a directed order as well, and thus determine a temporal direction. That is perfectly compatible with the argument of this paper. My conclusion is just that, in the context described above, the time order prescribed by any given observer in special relativity is not objective — it is not observer-independent.

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